Hilbert series of quasi-invariant polynomials in characteristics $p \leq n$

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Symmetric Polynomials

Symmetric polynomials stay the same after we permute its variables.

Example

In $\mathbf{k}[x_1, x_2, x_3]$:

1. 0
2. $x_1^2 + x_2^2 + x_3^2$
3. $x_1 x_2 x_3$

Symmetric Polynomials

S_n acts on $\mathbf{k}[x_1, x_2, ..., x_n]$ by permuting $(x_1, x_2, ..., x_n)$.

If s_{ij} is the map that swaps x_i and x_j, then P is symmetric if $s_{ij}P = P$ or $P - s_{ij}P = 0$.

Denote $(1 - s_{ij})P = P - s_{ij}P$.

Definition

Let $\mathbf{k}[x_1, x_2, ..., x_n]^{S_n}$ denotes the space of symmetric polynomials in $\mathbf{k}[x_1, x_2, ..., x_n]$.

Quasi-Invariants

Generalize symmetric polynomials?

Definition (quasi-invariants)

A polynomial P is m-**quasi-invariant** iff $(x_i - x_j)^{2m+1} | P - s_{ij}P$.
$Q_m(n, \mathbf{k})$ denotes the space of m-quasi-invariants in n variables over a field $\mathbf{k}$.

Remark

Symmetric polynomials are m-quasi-invariant for all m.

All polynomials are 0*-quasi-invariant.*

Examples of quasi-invariants

For 3 variables in characteristic 3,

$$(1 - s_{12})(x_1^3 - x_2^3) = 2(x_1^3 - x_2^3)$$
$$(1 - s_{13})(x_1^3 - x_2^3) = x_1^3 - x_3^3$$
$$(1 - s_{23})(x_1^3 - x_2^3) = x_3^3 - x_2^3$$

and $(x_i - x_j)^3 = x_i^3 - x_j^3$ over $\mathbf{F}_3$. Thus, $x_1^3 - x_2^3 \in Q_1(3, \mathbf{F}_3)$ is 1-quasi-invariant.

Question

Question

What does the structure of quasi-invariants look like?

Hilbert series

How to describe a vector space?

Definition

The Hilbert series of a vector space V of polynomials is the power series

$$\mathcal{H}(V) = \sum_d \dim V[d] t^d$$

where $V[d]$ is the subspace of V with degree d.

Goal

Compute the Hilbert Series of the m-quasi-invariants.

Hilbert series

Example

Let V be the polynomials in x over $\mathbf{C}$.

$$V[0] = \mathbf{C}$$
$$V[1] = \{cx | c \in \mathbf{C}\}$$
$$V[2] = \{cx^2 | c \in \mathbf{C}\}$$
$$\vdots$$

$\mathcal{H}(V) = \sum_d \dim V[d] t^d = \sum_d t^d = \frac{1}{1-t}.$

Generators + Modules

Definition

A module M over a ring R is a set equipped with an addition operation and scalar multiplication by elements of R.

A set S in M is a **generating set** if for all $m \in M$, there exists $r_s \in R$ such that $m = \sum_{s \in S} r_s s$.

Example

$Q_m(n, \mathbf{k})$ forms a module over $\mathbf{k}[x_1, ..., x_n]^{S_n}$.
If P is m-quasi-invariant and S is symmetric,

$$SP - s_{ij}(SP) = S(P - s_{ij}P)$$

which is divisible be $(x_i - x_j)^{2m+1}$

2 variable quasi-invariants

In 2 variables, the only condition for quasi-invariants is that $(x_1 - x_2)^{2m+1} | (1 - s_{12})P$.

Proposition

$Q_m(2, \mathbf{k})$ is generated by 1 and $(x_1 - x_2)^{2m+1}$ over $\mathbf{k}[x_1, ..., x_n]^{S_n}$.

Remark

The following proof only works for characteristic $p > 2$.

K is symmetric

Let $(1 - s_{12})P = (x_1 - x_2)^{2m+1}K$.

K is symmetric since

$$s_{12}\left((x_1 - x_2)^{2m+1}K\right) = -(x_1 - x_2)^{2m+1}s_{12}K$$
$$= -(x_1 - x_2)^{2m+1}K.$$

2 variable quasi-invariants

Since K is symmetric $s_{12}\left((x_1 - x_2)^{2m+1}K\right) = -(x_1 - x_2)^{2m+1}K$.

Idea

For some polynomial L, if $s_{12}L = -L$ then $(1 - s_{12})L = 2L$.

Then

$$(1 - s_{12})P = (1 - s_{12})\left(\frac{1}{2}(x_1 - x_2)^{2m+1}K\right)$$

implies

$$(1 - s_{12})\left(P - (x_1 - x_2)^{2m+1}\frac{K}{2}\right) = 0.$$

2 variable quasi-invariants

If

$$(1 - s_{12}) \left(P - (x_1 - x_2)^{2m+1} \frac{K}{2} \right) = 0$$

then $P - (x_1 - x_2)^{2m+1} \frac{K}{2}$ is symmetric.

$$\Rightarrow P = S + (x_1 - x_2)^{2m+1} \frac{K}{2}$$

for some symmetric polynomial S, so P is generated by 1 and $(x_1 - x_2)^{2m+1}$

Quasi-invariants in characteristic 0?

Theorem (Felder and Veselov, 2003)

$$\mathcal{H}\left(Q_m(3,\mathbf{C})\right) = \frac{t^0 + 2t^{3m+1} + 2t^{3m+2} + t^{6m+3}}{(1-t)(1-t^2)(1-t^3)}$$

This implies $Q_m(3,\mathbf{C})$ is freely generated by generators in degree 0, $3m+1$, $3m+2$, $6m+3$ as a $\mathbf{C}[x_1,x_2,x_3]^{S_3}-$module.

Remark

Degree 0 *generator is* 1 *and generates symmetric polynomials, the rest generates nonsymmetric.*

Characteristic p

Most of the time the Hilbert series of m-quasi-invariants in characteristic p is the same as that characteristic 0.

Characteristic p

Theorem (Ren and Xu, 2020)

Let $m \geq 0$ and $n \geq 3$ be integers. Let p be a prime such that there exists integers $a \geq 0$ and $k \geq 0$ with

$$\frac{mn(n-2) + \binom{n}{2}}{n(n-2)k + \binom{n}{2} - 1} \leq p^a \leq \frac{mn}{nk+1}.$$

Then the Hilbert series of $Q_m(n, \mathbf{C})$ is different from the Hilbert series of $Q_m(n, \mathbf{F}_p)$.

Characteristic p

To prove it, Ren and Xu found an explicit nonsymmetric polynomial in $Q_m(3, \mathbf{F}_p)$ in degree less than the smallest nonsymmetric polynomial in $Q_m(3, \mathbf{C})$.

Example

$(x_1 - x_2)^3$ is in $Q_1(3, \mathbf{F}_3)$. At the same time, $Q_1(3, \mathbf{C})$ has generators in degree 0, 4, 5 and 9.

Characteristic p

Goal:

To prove the cases Ren and Xu found are the only times the Hilbert series of $Q_m(3, \mathbf{F}_p)$ differs from that of $Q_m(3, \mathbf{C})$.

Characteristic $p > 3$

Theorem (Wang, 2023)

When $p > 3$,

$$\mathcal{H}\left(Q_m(3, \mathbf{F}_p)\right) = \frac{1 + 2t^{3m+1-d} + 2t^{3m+2+d} + t^{6m+3}}{(1-t)(1-t^2)(1-t^3)}$$

for some positive integer d.

What about for $p \leq 3$?

Representations and our problem:

- If p does not divide $|S_3|$ Maschke's Theorem states all representations in $\mathbf{F}_p$ can be decomposed into irreducible representations.
- Wang used Maschkes to prove the results for $p > 3$.
- Maschke's theorem no longer holds when $p||S_3|$.
- If $p = 2$ or 3, we must consider **modular representation theory** instead.

Extending

Our Project

What is the structure of the m-quasi-invariants in characteristic 2 and 3?

Quasi-invariants in characteristic 2

Turns out we can describe the generators of $Q_m(3, \mathbf{F}_2)$ explicitly.

Theorem

Let a be the largest natural number such that $2^a \leq 2m+1$.
The generators for $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ *are*

$$1,$$

$$(x_1 - x_2)^{2^{a+1}},$$

$$(x_1 - x_2)^{2^a} \prod (x_i - x_j)^{2m+1-2^a},$$

and

$$(x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2) \prod (x_i - x_j)^{2m}.$$

Quasi-invariants in characteristic 3

Theorem

If d is the degree of Ren and Xu's explicit nonsymmetric polynomial in $Q_m(3, \mathbf{F}_3)$ if it exists, and $3m+1$ otherwise,

$$\mathcal{H}\left(Q_m(3, \mathbf{F}_3)\right) = \frac{t^0 + 2t^d + 2t^{6m+3-d} + t^{6m+3}}{(1-t)(1-t^2)(1-t^3)}.$$

Conclusion

In this project we have explicitly described the Hilbert series of m-quasi-invariants in 3 variables in characteristic $p \leq 3$.

This brings us close to describing the Hilbert series of m-quasi-invariants in 3 variables for all characteristics.

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